Chapter 1

Equation Homogeneity
AICE Standard: 1.2c

Homogeneous Equations

- Equations in science for them to have any sense in meaning, they must be homogeneous.
- **An equation is considered to be homogeneous if and only if the units on the left-hand-side (LHS) are equal to the units on the right-hand-side (RHS).**
- An equation that is homogeneous is considered to be dimensionally correct because units also are a measurement of dimension.
- Think of homogeneity as a consequence of equations having to be consistent.

Dimensions in Equations

- **Dimensions in equations are properties pertaining to the SI *BASE* units.**
  - Common dimensions are time, distance, temperature, mass, current… see a trend? It's the quantities associated with the base units.
- Example: $x = vt$ in dimensional form is $L = L\ t^{-1}\ t$ where x has dimension L, v has dimensions $L\ t^{-1}$ and t has dimension t. We see the t disappears and we are left with $L = L$. Therefore it is dimensionally correct and therefore homogeneous.

Dimensions in Equations: Disclaimer

- Dimensional form of equations is **NOT** tested on the Cambridge Exam and I will **NOT** test you on it either.
- Dimensional form is just an alternative means of verifying homogeneity of equations and serves as a prerequisite for the procedure we use in this course.
- So for Example: Example: $x = vt$ in dimensional form is $L = L\,t^{-1}\,t$ where x has dimension L, v has dimensions $L\,t^{-1}$ and t has dimension t. **This notation is NOT on the Cambridge Exam!**

Proving Homogeneity of Equations

- Let’s refer back to x = vt. We have proven both sides of the equation have the same dimension. **Each variable corresponds to a dimension or a set of dimensions. Like v for example has dimensions L t^{-1} since speed is DISTANCE DIVIDED BY TIME.**
- Let’s prove that x = vt is homogeneous.
    - x is measured in meters (m), v is measured in meters per second (ms^{-1}) and t is measured in seconds (s).
    - Thus, m = m s^{-1} s.
    - Note the s^{-1} and s cancel per exponent rules and we left with m = m therefore the equation is homogeneous!

Homogeneity and Base Units

- Proving homogeneity of equations does require you to convert all units into base units.
- That does mean that derived units must be broken into their base forms such that you only see the special seven base units.
- **One can think of equation homogeneity as “balancing equations” from chemistry but we are simply balancing units rather than atoms of elements.**

Other Terminology and Nuances

- **A quantity or variable is dimensionless if it can be represented in such a manner where the SI base units cancel out.**
    - The easiest and probably most humorous example of this is the degree and radian which is a unit of angular measure.
    - You might be thinking, Mr. Grenda, that's absurd. *Degrees and radians are units, but the unit itself is dimensionless.*
    - Proof that the radian is dimensionless.
        - Define the radian as the ratio of arc length to radius therefore we have $s / r = \phi$. Arc length is a measurement of distance so it has units m, and radius is a measurement of distance so it has units m.
        - m/m = 1. Weird right? Degrees and Radians are dimensionless.
- **Functions such as sin, cosine, e^x do not have dimensions.**
- **Numbers and constants are dimensionless, this includes stuff such as pi too.**

Example Problem: Classic Wave Equation

- $x(t) = A\sin(kx + \omega t + \Phi)$
- The statement below is the full classic wave equation where:
  - $x(t)$ is position which measures distance.
  - A is amplitude which measures distance.
  - k is the wave-number which measures inverse distance.
  - x is the horizontal position which measures distance.
  - ω is the angular frequency which is the rate of change of the angle.
  - t is time
  - Φ is the phase which is measured in radians.

Example Problem: Classic Wave Equation

- $x(t) = A\sin(kx + \omega t + \Phi)$
- The statement below is the full classic equation where:
    - $x(t)$ is position which measures distance
    - A is amplitude which measures distance
    - k is the wave-number which measures inverse distance
    - x is the horizontal position which measures distance
    - ω is the angular frequency which is the rate of change of the angle.
    - t is time
    - Φ is the phase which is measured in radians.
- So if the equation is homogeneous the entire RHS must simplify to meters (m) because LHS is measured in meters.
    - $m = m (m^{-1} m + s^{-1} s)$ because the angular frequency and phase have a radian component which does not contribute we are left with the statement above which simplifies to m = m so the equation is homogeneous.

Example Problem 1

- Show that the formula $x = v_t t + 0.5at^2$ is homogeneous.
- We immediately see from the left hand side that x is by itself with units of m so the right hand side must simplify to m.
    - $v_i t$ simplifies to $ms^{-1}s$ so it simplifies to m.
    - $0.5at^2$ simplifies to $m\ s^{-2}\ s^2$ where 0.5 has no units and we see it too simplifies to m.
    - Thus we have m = m + m or m = 2m and we can throw away the 2.
    - Hence m = m.

Example Problem 2

- Show that the equation for the Hall Voltage is homogeneous. $V_H = BI/ntq$
  - Where V_H is the Hall Voltage, B is the magnetic field, I is the current, n is charge density, q is charge and t is thickness.
- The base unit of voltage is $kg\ m^2\ s^{-3}\ A^{-1}$ so BI/nqt must be equal to this expression.
- B is $kg\ s^{-2}\ A^{-1}$, I is A, nq is $As\ m^{-3}$ t is m.
- Plugging in we obtain $kg\ m^2\ s^{-3}\ A^{-1} = (kg\ s^{-2}\ A^{-1})(A) / (As\ m^{-3})(m)$
- Simplifying we obtain $kg\ m^2\ s^{-3}\ A^{-1} = kg\ s^{-2} / A\ s\ m^{-2}$ which through algebra yields the left hand side. We can apply $1/x = x^{-1}$ to the denominator and then simplify.

Solving for a Mystery Quantity

- Sometimes Cambridge likes to modify how these questions are asked and they give you an equation and say it is homogeneous and ask you to determine the quantity of exponent for a variable that makes it homogeneous.
  - For example: What value of n makes the following equation homogeneous: $a_c = v^n/r$ where a is centripetal acceleration, v is speed and r is radius of the circular arc.

Example Problem: Solving for a Mystery Quantity

- What value of n makes the following equation homogeneous: $a_c = v^n/r$ where a is centripetal acceleration, v is speed and r is radius of the circular arc.
  - Acceleration is the second rate of change of position so it has units of ms^{-2} and speed is ms^{-1} and radius is m.
  - Thus we have $ms^{-2} = (ms^{-1})^n / m$ because are are attempting to figure out n and n operates on v, v^n. So what exponent satisfies the equation such that it is homogeneous?
  - We see n = 2, and we can check our result: $ms^{-2} = (ms^{-1})^2\ m^{-1}$. Our m in the denominator, I brought it up as m^{-1}. Applying exponent rules we see:
    - $ms^{-2} = m^2s^{-2}\ m^{-1}$ therefore $ms^{-2} = ms^{-2}$.

Disclaimer About These Problems

- These problems are tricky at the beginning because we are not familiar with the units of many of these things but as you acquire knowledge about units as we progress through the course it will be much easier.
- As an exercise, I suggest verifying every equation we come across in this course is homogeneous to build practice and familiarize ourselves with the base units and derived units.

Pitfall Prevention: Unit Placement and Powers

- The easiest way to get messed up with these problems is not realizing the units themselves follow algebra rules. Here is an example demonstrating this:
- Show that $T = 2\pi \sqrt{(m / k)}$
- T is period of oscillation which is a measurement of time, m is the mass of the oscillating object and k is the spring constant.
- **Show means prove or derive.**

Pitfall Prevention: Unit Placement and Powers

- $T = 2\pi \sqrt{(m / k)}$
- T is period of oscillation which is a measurement of time, m is the mass of the oscillating object and k is the spring constant.
- We immediately see the LHS is measured in seconds so the RHS should reduce to seconds too!
- Let us go about demonstrating this.

Pitfall Prevention: Unit Placement and Powers

- $T = 2\pi \sqrt{(m / k)}$
- T is period of oscillation which is a measurement of time, m is the mass of the oscillating object and k is the spring constant. **Recall 2 and pi have no units!**
- $s = (kg / kg\ m\ s^{-2}\ m^{-1})^{½}$ is what it would look like before simplification since remember a square root is the same as raising something to the one-half power.
- Kg m s^{-2} m^{-1} are the base units of the spring constant.

Pitfall Prevention: Unit Placement and Powers

- $s = (\text{kg} / \text{kg m s}^{-2}\,\text{m}^{-1})^{½}$ is what it would look like before simplification since remember a square root is the same as raising something to the one-half power.
- We see the meter contribution and the kilogram contribution disappear easily so we are left with $s = (1/s^{-2})^{½}$. Recall that the s^{-2} can be written like $1/s^2$ so we have $1/1/s^2$ so we have a / b/c which we know can be simplified to be (ac/b) so we have $s = (s^2)^{½}$ and we see the root of one-half turns the 2 to a 1 due to exponent rules so we have s = s.
- The logic above is similar to that of x^{-2} can be written as $1/x^2$.
- This is an example where algebraic simplification makes the result more intuitive and easier to see.
- This is likely the most challenging thing Cambridge can do for this concept but it's good to be aware of this should it come up.

Example Problem 3

- Determine the power of n such that the formula is homogeneous.
  - $j = \sigma T^n$ where j is the black body radiance, σ is the Stefan Boltzmann constant and T is the absolute temperature.
  - Units:
    - $j = Wm^{-2}$
    - $\sigma = Wm^{-2}K^{-4}$
    - $T^n = K^n$
- We can see from inspection that n must be 4.
- Thus we have $Wm^{-2} = Wm^{-2}K^{-4}\ K^4$ thus both sides are equal.

Example Problem 4

- Determine the power of n such that the formula is homogeneous.
  - $P = 1/2\rho v^n$ where P is fluid pressure, ρ is fluid density and v is fluid velocity.
  - Units:
    - $P = kg\ m^{-1}\ s^{-2}$
    - $\rho = kg\ m^{-3}$
    - $v^n = (ms^{-1})^n$
  - Thus we have $kg\ m^{-1}\ s^{-2} = kg\ m^{-3}\ (ms^{-1})^n$ we see n must be 2 to balance the s^{-2} on the left hand side and also force m^2 and m^{-3} to become m^{-1}.

Closing Remarks about Equation Homogeneity

- This method of checking equations is incredibly powerful when checking your work in a physics or chemistry course.
- You can ALWAYS use this to determine if your algebra is correct, if you remembered an equation correctly, or again checking your work during a calculation or a proof/derivation/show problem.
- Do not underestimate the value of this property of equations. It exists everywhere.